1 Technical Details

1.1 Fitting volatility curve with SVI

1.1.1 General Problem

SVI parametric formulation (e.g. Zeliade, 2009) of the volatility Curve is

$$\sigma_{BS}^2(k) = a + b\left(\rho(k-m) + \sqrt{(k-m)^2+\sigma^2}\right)$$

where $k = \log(Strike/Forward)$ and (m, σ, ρ, a, b) – parameters. This functional form assumes $a \in \mathbb{R}, b \geq 0, |\rho| < 1, m \in \mathbb{R}, \sigma > 0$ and constraint

$$a + b\sigma\sqrt{1-\rho^2} \geq 0 \Rightarrow a \geq -b\sigma\sqrt{1-\rho^2}$$

that insures that this function lies above 0 everywhere. Absence of static arbitrage requires

$$b \leq \frac{4}{(1+|\rho|)T}$$

Zeliade notes that for large maturities, almost affine smiles are not uncommon. This corresponds to the case when $\sigma \to 0$ or when $\sigma \to \infty, a \to -\infty$. To rule out the first limiting case, Zeliade (2009) restricts $\sigma \geq \sigma_{min} > 0$. To rule this the second, Zeliade (2009) assumes that $a \geq 0$. However, I found that it is hard to fit the smile when $a \geq 0$, since implied variances for strikes close to current price are too close to zero, so that we may want to set parameter $a < 0$ which is in general doesn't violate anything.

1.1.2 Zeliade (2009) method of reducing dimensionality

The approach of Zeliade (2009) then minimizes sum of squared residuals over parameters $\theta = (m, \sigma, \rho, a, b)$

$$\min_\theta \sum_i \left[a + b\left(\rho(k_i - m) + \sqrt{(k_i-m)^2+\sigma^2}\right) - \sigma_i^2\right]^2$$

where k_i – observed $\log(Strike_i/Forward_i)$ and σ_i is the observed implied variance. If we follow Zeliade (2009) assumption that $a \geq 0$, then we can easily transform this problem into a linear one for fixed (m, σ):

1. Divide the parameter θ into two parts $\theta_1 = (m, \sigma)$ and $\theta_2 = (\rho, a, b)$. Fix (m, σ) and substitute $y_i = \frac{k_i - m}{\sigma}$ so that the minimization objective becomes

$$\sum_i \left[a + b\left(\rho\sigma y_i + \sigma\sqrt{y_i^2+1}\right) - \sigma_i^2\right]^2$$

2. Zeliade (2009) works with total variance Tv rather than on variance σ^2. Denote total variance $\tilde{v} = Tv$, so that the SVI becomes

$$v(k) = aT + bT\left(\rho(k-m) + \sqrt{(k-m)^2+\sigma^2}\right)$$

and the minimization objective becomes

$$\sum_i \left[aT + b\rho\sigma T y_i + b\sigma T\sqrt{y_i^2+1} - \tilde{v}_i^2 \right]^2$$

Replace variables $\tilde{a} := aT, d := b\rho\sigma T, c := b\sigma T$. Now the problem is just a linear least squares regression for the new variables

$$\sum_i \left[\tilde{a} + dy_i + c\sqrt{y_i^2+1} - \tilde{v}_i^2 \right]^2$$

3. Now we need to deal with constraints.

$$\rho \in [-1,1] \Rightarrow |d| \le c$$

$$b \ge 0 \Rightarrow c \ge 0$$

$$b \le \frac{4}{(1+|\rho|)T} \Rightarrow c \le \frac{4\sigma}{1+|\rho|} \Rightarrow c + c|\rho| \le 4\sigma \Rightarrow c + |d| \le 4\sigma \Rightarrow |d| \le 4\sigma - c$$

$$c \le \frac{4\sigma}{1+|\rho|} \Rightarrow c \le 4\sigma$$

$$0 \le a \le \max_i v_i \Rightarrow 0 \le \tilde{a} \le \max_i \tilde{v}_i$$

Thus, we can described the parameter space as

$$\mathcal{D} = \begin{cases} 0 \le c \le 4\sigma \\ |d| \le c, |d| \le 4\sigma - c \\ 0 \le \tilde{a} \le \max_i \tilde{v}_i \end{cases}$$

1.1.3 Simplification of Berger, Dew-Becker and Giglio

Berger, Dew-Becker and Giglio (?) assumes that $\rho = 0$. ρ controls the assymetry of asymptotes of a hyperbola and thus asymmetry of the slopes of wings of the volatility smile. They say that this including this ρ has a minimal effect on the fit. In this case the smile positivity condition simplifies to $a \ge -b\sigma \Rightarrow \tilde{a} \ge -c$. In this case, the optimization simplifies the following procedure

1. For fixed (m, σ) the objective becomes

$$\min_{\tilde{a},d} \sum_i \left[\tilde{a} + c\sqrt{y_i^2+1} - \tilde{v}_i^2 \right]^2$$

subject to

$$\mathcal{D} = \begin{cases} 0 \le c \le 4\sigma \\ -c \le \tilde{a} \le \max_i \tilde{v}_i \end{cases}$$

$\mathcal{D}$ defines a parallelogram in the parameter space and minimization objective is a convex function.

2. Define

$$X = \begin{pmatrix} 1 & \sqrt{y_1 + 1} \\ \vdots & \vdots \\ 1 & \sqrt{y_n + 1} \end{pmatrix}, \tilde{v} = \begin{pmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{pmatrix}$$

- Estimate linear regression $\beta := (\tilde{a}\ c)' = (X'X)^{-1}X'\tilde{v}$. If $\beta \in \mathcal{D}$ then we found the minimum. If $\beta \notin \mathcal{D}$ proceed further
- Estimate regression along the side of domain $\mathcal{D}$. Under a linear constraint on parameters $R\beta = b$, $\beta = \arg\min(X\beta - \tilde{v})'(X\beta - \tilde{v})$ is given by

$$\beta = (X'X)^{-1}(X'\tilde{v} + R'\lambda) \text{ where } \lambda = \left[R(X'X)^{-1}R'\right]^{-1}\left[b - R(X'X)^{-1}X'\tilde{v}\right]$$

  Linear constraints for sides of $\mathcal{D}$ are

$$\begin{aligned} (c = 0) &: R = (0\ 1), b = 0 \\ (c = 4\sigma) &: R = (0\ 1), b = 4\sigma \\ (\tilde{a} = -c) &: R = (1\ 1), b = 0 \\ (\tilde{a} = \max_i \tilde{v}_i) &: R = (1\ 0), b = \max_i \tilde{v}_i \end{aligned}$$

  For each of the constraints we need to check that the solution satisfies all other inequalities. If it doesn't, it can't be a solution candidate
- Estimate objective in 4 vertices

$$\begin{aligned} &\tilde{a} = 0, c = 0 \\ &\tilde{a} = -4\sigma, c = 4\sigma \\ &\tilde{a} = \max_i \tilde{v}_i, c = 0 \\ &\tilde{a} = \max_i \tilde{v}_i, c = 4\sigma \end{aligned}$$

- Pick the solution along the sides and vertices that has the lowest objective.

Relaxing the constraint from $\tilde{a} \geq 0$ to $\tilde{a} \geq -c$ seems to improve the fit from visual inspection.

This is one of the methods that I use to fit the volatility smile. The main problem as it is also outline in Berger, Dew-Becker and Giglio (?) is that the minimization is very sensitive to starting values and hence they use grid search over $m \times \sigma = [-1, 1] \times [0.00001, 10]$ to pick a starting value and then they use local derivative free minimization algorithm. I do the same in one of the approaches. In another approach, I use global optimization algorithm. These algorithms require the set to be bounded and I use the same bounds as in the grid search. I then compare both of these approaches by plotting fit of each.

1.1.4 Proceeding without assuming $\rho = 0$ or $\tilde{a} \geq 0$

Zeliade (2009) doesn't assume $\rho = 0$ and they can still derive a simple objective. The reason is that they assume $\tilde{a} \geq 0$ in which case the constraint becomes linear. If we don't assume this, we are left with constraint $\tilde{a} \geq -c\sigma\sqrt{1 - \rho^2}$ so that we can't use the same bounded linear regression approach. It seems that the only way to proceed is to perform factor out a and b out

of optimization and perform the outer numerical optimization over (m, σ, ρ). The issue is that we either need to add the third dimension to the grid search or perform global optimization over 3 variables. Grid search over 3 variables is quite costly. Global optimization also adds computational complexity but not to the same extent. Nevertheless, I use the two approaches in order to compare the fit and see if it is worthwhile to fit volatility smile in this way.

For fixed (m, σ, ρ) the minimization objective becomes

$$\min_{a,b} \sum_i \left[a + b\left(\rho\sigma y_i + \sigma\sqrt{y_i^2+1}\right) - \sigma_i^2 \right]^2 \text{ subject to}$$

$$(a,b) \in \mathcal{D} = \begin{cases} -b\sigma\sqrt{1-\rho^2} \leq a \leq \max_i\{\sigma_i\} \\ 0 \leq b \leq \dfrac{4}{(1+|\rho|)T} \end{cases}$$

that is very similar to the problem considered before.

1.2 Generating Option Prices

1.2.1 IvyDB approach to implied volatility

In order to calculate option price for given strike and a given volatility we need to know the Zero Coupon Rate for the maturity of the option and the type of dividends. The approach for calculating option prices for indices and individual equities differs (I took details from IvyDB reference on WRDS (link)):

1. For indices IvyDB assumes that dividends are continuously compounded. They use a structural regression that utilizes put-call parity to get dividends yield. Table OPTIONM.IDXDVD provides such estimate for the dividend yield.

2. For equities IvyDB uses either actual ex-dividend date if dividends were announced or it uses projected date using the frequency of dividends. Information about historical dividends and other distributions is provided in table OPTIONM.DISTRD

Zero Coupon Rates for standard maturities is provided in Table OPTIONM.ZEROCD where IvyDB provides annualized continuously compounded interest rate. To get the interest rate for the maturity of the option IvyDB linearly interpolates interest rates for neighboring maturities.

1.2.2 Continuously Compounded vs. Discrete Dividends

Value of an option with continuously compounded dividends at rate q is

$$\text{Call Price} = e^{-qT}\mathcal{N}(d_1) - e^{-rT}K\mathcal{N}(d_2)$$

$$\text{Put Price} = e^{-rT}K\mathcal{N}(-d_2) - e^{-qT}\mathcal{N}(-d_1)$$

$$\text{where } d_1 = \frac{\log(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

Note that in the case of continuously compounded dividends forward price at time 0 with maturity at time T of an asset is given by

$$F_0(T) = \frac{e^{-qT}S_0}{P_0(T)} = \frac{e^{-qT}S_0}{e^{-rT}}$$

where $P_0(T)$ is the price of a ZCB.

In the case of discrete dividends between current time 0 and the maturity of the option T at points of time $\tau_1, \ldots, \tau_N$ of size $D_1, \ldots, D_N$

$$F_0(T) = \frac{S_0 - \sum_i P_0(\tau_i)D_i}{P_0(T)}$$

To value an option for such asset we can use the formula from above replacing q with 0 and S_0 with $F_0(T)P_0(T)$ (see for example, Back (2017)). For the case of continuously compounded dividends $F_0(T)P_0(T) = e^{-qT}S_0$. Therefore, both continuously compounded and discrete dividends are cases of a more general formula. Option prices are given by standard Black-Scholes Formula with no dividends

$$\text{Call Price} = \mathcal{N}(d_1) - e^{-rT}K\mathcal{N}(d_2)$$

$$\text{Put Price} = e^{-rT}K\mathcal{N}(-d_2) - \mathcal{N}(-d_1)$$

$$\text{where } d_1 = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

where we replace S_0 with $F_0(T)P_0(T)$ and $F_0(T)$ is calculated as

1. in the case of continuously compounded dividednds as $\frac{e^{-qT}S_0}{P_0(T)}$
2. in the case of discrete dividends as $\frac{S_0 - \sum_i P_0(\tau_i)D_i}{P_0(T)}$

To get price of ZCB I use a table with ZCB rates from IvyDB and linearly interpolate them, if dividend is paid before the minimal maturity in the ZCB table I set the rate equal to the one for minimal maturity.

1.3 Calculating Du and Kapadia Measure of Jump Risk

1.3.1 Comment about $\mu_{0,T}$ term

Derivation of $\mathbb{D}(T)$ term require calculating

$$\mu_{0,T} = E_0^* \left[\log \frac{S_T}{S_0} \right]$$

In the paper they do the following. First, denote $R_T = \log(S_T/S_0)$. Second, using the martingale property under the risk neutral measure they write

$$e^{rT} = E_0^* \left[\frac{S_T}{S_0} \right] = E_0^* \left[e^{R_T} \right]$$

Second, expand function e^R around $R = 0$ to get

$$e^{rT} = E_0^* \left[1 + R_T + \frac{1}{2} R_T^2 + \frac{1}{6} R_T^3 + \frac{1}{24} R_T^4 \right]$$

$$e^{rT} = 1 + E_0^* R_T + \frac{1}{2} E_0^* R_T^2 + \frac{1}{6} E_0^* R_T^3 + \frac{1}{24} E_0^* R_T^4$$

$$E_0^* \left[\log \frac{S_T}{S_0} \right] \equiv E_0^* R_T = e^{rT} - 1 - \frac{1}{2} E_0^* R_T^2 - \frac{1}{6} E_0^* R_T^3 - \frac{1}{24} E_0^* R_T^4$$

Third, they use option spanning theorem that any contract with twice continuously differentiable payoff function $H(S)$ has price

$$e^{-rT} E_0^*[H(S)] = e^{-rT}(H(\bar{S}) - \bar{S} H'(\bar{S})) + H'(\bar{S}) S_0 + \int_{\bar{S}}^{\infty} H''(K) C(0,T,K) dK + \int_0^{\bar{S}} H''(K) P(0,T,K) dK$$

to price each of the terms $E_0^* R_T^2$, $E_0^* R_T^3$ and $E_0^* R_T^4$.

If we let

$$H(S_T) = \log \frac{S_T}{S_0} = \log(S_T) - \log(S_0)$$

we can write

$$e^{-rT} E_0^*[\log(S_T)] = e^{-rT}(\log(\bar{S}) - \bar{S} \frac{1}{\bar{S}}) + \frac{1}{\bar{S}} S_0 - \int_{\bar{S}}^{\infty} \frac{1}{K^2} C(0,T,K) dK - \int_0^{\bar{S}} \frac{1}{K^2} P(0,T,K) dK$$

1.4 Regression with linear constraints on parameters

The problem is

$$\min_{\beta} \frac{1}{2}(X\beta - \tilde{v})'(X\beta - \tilde{v}) \text{ subject to } R\beta = b$$

Set up lagrangian

$$\mathcal{L} = \frac{1}{2}(X\beta - \tilde{v})'(X\beta - \tilde{v}) - \lambda'(R\beta - b)$$

First order condition

$$\frac{\partial \mathcal{L}}{\partial \beta} = (X\beta - \tilde{v})'X - \lambda' R = 0 \Rightarrow \beta' X'X - \tilde{v}'X - \lambda' R = 0 \Rightarrow \beta = (X'X)^{-1}(X'\tilde{v} + R'\lambda)$$

Plug this into constraint to get

$$R(X'X)^{-1}(X'\tilde{v} + R'\lambda) = b \Rightarrow R(X'X)^{-1}X'\tilde{v} + R(X'X)^{-1}R'\lambda = b \Rightarrow$$

$$\lambda = \left[R(X'X)^{-1}R'\right]^{-1}\left[b - R(X'X)^{-1}X'\tilde{v}\right]$$

If we plug λ back into the expression for β we can get the final answer.